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The Effect of Preprocessing to the Complexity of List Sphere Detector Algorithms

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ABSTRACT

A list sphere detector (LSD) is an enhancement of a sphere detector (SD) that can be used to approximate the soft output MAP detector used in the detection of the multiple-input multiple-output (MIMO) signals. The LSD algorithm executes a tree search on the given lattice and returns a candidate list. The LSD algorithm complexity, i.e., the number of visited nodes in the search tree, can be decreased by applying proper ordering of the transmitted spatial streams in the detection. In this paper, we study the effect of two sophisticated preprocessing methods, the channel matrix column ordering based on Euclidean norm and the sorted QR decomposition (SQRD), to the performance and complexity of the LSD algorithms and compare them to the traditional QR decomposition (QRD). We show that the SQRD preprocessing is a simple way to decrease complexity of the LSD and it decreases the number of visited nodes approximately $20-30\%$ compared to the QRD which results in significant number of saved arithmetic operations in the LSD. We also show that the plain channel matrix column ordering is not feasible preprocessing method to be used with LSD in highly correlated channel realization.

I. Introduction

The ever increasing data rates in wireless communication systems require the use of the available bandwidth as efficiently as possible to maximize the capacity of the system. Orthogonal frequency division multiplexing (OFDM) [1] has become a widely used technique to significantly reduce receiver complexity in broadband wireless systems. Multiple-input multiple-output (MIMO) channels offer improved capacity and significant potential for improved reliability compared to single antenna channels [2]. The MIMO concept in combination with OFDM (MIMO-OFDM) has been adapted to multiple wireless telecommunication standards, such as the 3rd generation partnership project (3GPP) long term evolution (LTE) and IEEE 802.16e.

The optimal detector for a spatially multiplexed MIMO-OFDM signal without forward error coding (FEC) is the hard output maximum likelihood (ML) detector. Sphere detector (SD) calculates the hard output maximum likelihood (ML) solution with reduced complexity compared to full-complexity ML detectors [3], [4]. The optimal joint detection and decoding of a MIMO signal with FEC can be approximated with an iterative (turbo type) receiver with separate detector and decoder [5], where the optimal soft output detector is the maximum *a posteriori* probability (MAP) detector. However, the computational complexity of the MAP detector is an exponential function of the number of transmit antennas and modulation levels, and, thus, it is not typically promising in practical implementation. A list sphere detector (LSD) [5] is a variant of the sphere detector that can be used to approximate MAP detector with much lower computational complexity [5], [6], [7]. Depending on the list size, the LSD provides a tradeoff between the performance and the computational complexity.

The SD and LSD algorithms perform a closet point tree search in a lattice formed by the received signal vector. The number of studied nodes in the search tree in the sphere search is dependent on the applied algorithm and the channel realization. It has been shown that the SD algorithm complexity, i.e., the number of visited nodes, can be decreased by applying proper preprocessing of the detection order [4], [8], [9], [10]. The preprocessing of the channel matrix has to be recalculated as the channel changes, i.e., it is relative to the channel coherence time. Thus, the complexity reduction of the SD algorithm is obtained with much lower effort as the SD algorithm operates at symbol rate, which is typically much higher than channel coherence time. In this paper, we study the effect of different preprocessing methods to the complexity of LSD algorithms. We consider the traditional QR decomposition (QRD), the ordering of the channel matrix columns according to the norm, and the sorted QRD (SQRD) [11] as preprocessing before the LSD algorithm tree search and compare the different methods. We also show that the correlation properties of the channel have a major impact on the complexity of the LSD algorithm.

The paper is organized as follows. The signal model and the sphere detection principles are presented in Section II. The list sphere detector basic architecture, the considered preprocessing and LSD algorithms are introduced in Section III. The performance examples with introduced methods are presented and discussed in Section IV. Finally the conclusions are drawn in Section V.

II. MIMO Signal and Detection

An OFDM based multiple-antenna system with N_T transmit (TX) antennas and N_R receive (RX) antennas is considered with assumption $N_R \geq N_T$ and QAM constellation. The received signal at baseband can be expressed in terms of code

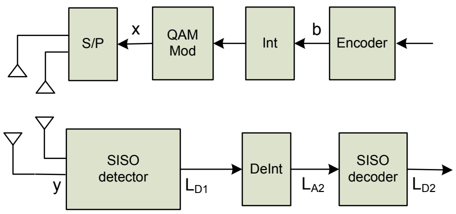

Fig. 1. A coded MIMO system model.

symbol interval as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \boldsymbol{\eta}, \tag{1}$$

where the received signal vector $\mathbf{y} \in \mathbb{C}^{N_R \times 1}$, the transmit symbol vector $\mathbf{x} \in \Omega^{N_T} \subset \mathbb{C}^{N_T \times 1}$ and the noise vector $\boldsymbol{\eta} \in \mathbb{C}^{N_R \times 1}$ are defined in the frequency domain. The elements of $\boldsymbol{\eta}$ are independent and complex zero-mean Gaussian with equal power σ^2 for both real and imaginary parts and represent the frequency domain thermal noise at the receiver. The channel matrix $\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$ contains complex Gaussian fading coefficients with unit variance. The entries of $\mathbf{x}$ are chosen independently from a complex QAM constellation Ω with Q bits per symbol. The complex system model in (1) can be reduced into an equivalent real model as follows

$$\begin{bmatrix} \mathrm{Re}(\mathbf{r}) \\ \mathrm{Im}(\mathbf{r}) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\mathbf{H}) & -\mathrm{Im}(\mathbf{H}) \\ \mathrm{Im}(\mathbf{H}) & \mathrm{Re}(\mathbf{H}) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\mathbf{x}) \\ \mathrm{Im}(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} \mathrm{Re}(\eta) \\ \mathrm{Im}(\eta) \end{bmatrix}. \tag{2}$$

Let us define the new real dimensions $M_\mathrm{T} = 2N_\mathrm{T}$, $M_\mathrm{R} = 2N_\mathrm{R}$. The real symbol alphabet is now $\Omega_R \subset \mathbb{Z}$, e.g., $\Omega_R = \{-3, -1, 1, 3\}$ in the case of 16-QAM.

We assume a practical case of system with forward error coding (FEC) and with separate soft-input soft-output (SISO) detector and decoder at the receiver as shown in Figure 1. The detector generates soft output information $L_{\mathrm{D}1}(b_k)$ of each transmitted bit b_k [5].

A. Sphere Detection

The sphere detectors (SDs) achieve the hard output maximum likelihood (ML) solution of $\mathbf{x}$ with a reduced number of considered candidate symbol vectors in the search compared to traditional exhaustive search algorithms. Then the sphere search is done by limiting the search to points that lie inside a M_R-dimensional hyper-sphere $S(\mathbf{y}, \sqrt{C_0})$ centered at $\mathbf{y}$. After QR decomposition (QRD) of the channel matrix $\mathbf{H}$, the condition can be written as [4]

$$||\tilde{\mathbf{y}} - \mathbf{R}\mathbf{x}||_2^2 \leq C_0, \tag{3}$$

where C_0 is the squared radius of the sphere, $\mathbf{R} \in \mathbb{R}^{M_R \times M_T}$ is an upper triangular matrix with positive diagonal elements, $\mathbf{Q} \in \mathbb{R}^{M_R \times M_R}$ is an orthogonal matrix, and $\tilde{\mathbf{y}} = \mathbf{Q}^\mathrm{H}\mathbf{y}$.

Due to the upper triangular form of $\mathbf{R}$ the values of $\mathbf{x}$ can be solved from (3) level by level using the back-substitution algorithm. Let $\mathbf{x}_i^{M_T} = (x_i, x_{i+1}, \ldots, x_{M_T})^\mathrm{T}$ denote the last $M_T - i + 1$ components of the vector $\mathbf{x}$. The squared partial Euclidean distance (PED) of $\mathbf{x}_i^{M_T}$ can be calculated as [6]

$$\begin{aligned} d(\mathbf{x}_i^{M_T}) &= d(\mathbf{x}_{i+1}^{M_T}) + |\tilde{y}_i - \sum_{j=i}^{M_T} R_{i,j}x_j|^2 \\ &= d(\mathbf{x}_{i+1}^{M_T}) + |b_{i+1}(\mathbf{x}_{i+1}^{M_T}) - R_{i,i}x_i|^2, \end{aligned} \tag{4}$$

where $d(\mathbf{x}_{M_T}^{M_T}) = 0$, $b_{i+1}(\mathbf{x}_{i+1}^{M_T}) = \tilde{y}_i - \sum_{j=i+1}^{M_T} R_{i,j}x_j$, $R_{i,j}$ is the (i,j)th term of $\mathbf{R}$ and $i = M_T, \ldots, 1$. Depending on the search strategy and the channel realization, the SD searches a variable number of nodes in the tree structure, and aims to find the point $\mathbf{x} = \mathbf{x}_1^{M_T}$, also called a leaf node, for which the Euclidean distance (ED) $d(\mathbf{x}_1^{M_T})$ is minimum.

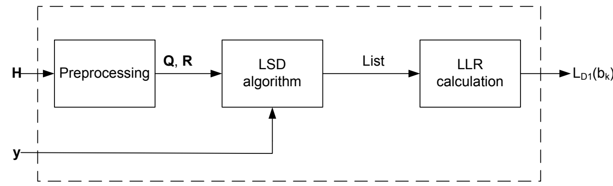

Fig. 2. A high level architecture of a list sphere detector.

III. List sphere detector

The SD algorithms give the ML solution as an output. However, the performance of a channel coded system may suffer significantly with ML detector compared to the optimal MAP detector. The list sphere detector (LSD) [5] can be used for obtaining a list of the most probable candidate symbol vectors $\mathcal{L} \in \mathbb{Z}^{N_\text{cand} \times N_\mathrm{T}}$ as an output, where N_cand is the size of the candidate list so that $1 \leq N_\text{cand} \leq 2^{QN_\mathrm{T}}$. The list can then be used to approximate the MAP solution with reduced complexity. Depending on the list size N_cand, it provides a tradeoff between the performance and the computational complexity. The inaccurate approximation can be compensated for by limiting the dynamic range of the output LLR variable [12].

A high level architecture of the list sphere detector structure is shown in Figure 2. The LSD architecture consists of the preprocessing unit, the LSD algorithm unit and the LLR calculation unit. The preprocessing unit decomposes the channel matrix $\mathbf{H}$ into upper triangular form as in (3), which enables the symbol-by-symbol tree search. The LSD algorithm unit executes the tree search and gives the candidate list $\mathcal{L}$ as an output. The number of visited nodes by the algorithm, which corresponds to the complexity of the algorithm, is dependent on the applied search strategy. The approximation of $L_\mathrm{D}(b_k)$ is calculated in the log-likelihood ratio (LLR) calculation unit using the given candidate list, and it can be implemented e.g. using the well-known Jacobian algorithm and a small look-up table [13].

A. Preprocessing methods

The preprocessing unit is used to decompose the channel matrix $\mathbf{H}$ into upper triangular form as in (3), which enables the symbol-by-symbol tree search with back substitution algorithm. Typically QRD is assumed in literature to perform the channel matrix decomposition into an upper triangular matrix

R and an orthogonal matrix **Q**, which are given as an input with received signal **y** to the LSD algorithm. However, it has been shown that the complexity of the SD algorithm search can be decreased by applying different more sophisticated ordering or preprocessing approaches before the SD algorithm [4], [8], [9], [10]. The preprocessing of the channel matrix has to be recalculated as the channel changes, i.e., it is relative to the channel coherence time. Thus, the complexity reduction of the SD algorithm is obtained with much lower effort as the SD algorithm operates at symbol rate, which is typically much higher than channel coherence time. Obviously one would also think that the complexity of the LSD algorithms can be decreased by similar approaches. In this paper, we consider two ordering methods for the channel matrix **H** and study their effect to the complexity and performance.

1) Column ordering based on Euclidean norm: The column ordering according to the Euclidean norm has been proposed for SD e.g. in [4], [9], [14]. In this method the channel matrix columns $\mathbf{h}_i$ are ordered in descending order according to the Euclidean norm $\|\mathbf{h}_i\|$ before the QRD, i.e., the signal from transmit antenna i with strongest channel gain $\|\mathbf{h}_i\|$ is ordered to be at the root layer of the search tree. This typically decreases the sphere search as the strongest signal decisions are made at the beginning of the tree traversal.

2) Sorted QRD: The sorted QRD (SQRD) [11] is an extension to the modification Gram-Schmidt procedure by reordering the columns of the channel matrix prior each orthogonalization step. The algorithm jointly calculates a very close to optimized detection order, which is achieved by the V-BLAST detection algorithm [15], and the QRD of the channel matrix. This means that the absolute values of the diagonal elements $|R_{i,i}|$ of the resulting upper triangular matrix **R** are minimized in the process of calculating the QRD. Thus, the strongest layer is located at the root layer of the search tree.

B. LSD algorithms

The list sphere detector algorithms can often be composed from the sphere detector algorithms with minor modifications. In this paper, we consider three different LSD algorithms based on different search strategies, the K-best-LSD [16], [17], the Schnorr Euchner enumeration (SEE) - LSD [18], [17], and the Increasing Radius (IR) - LSD [19], [20], [21]. The LSD algorithms were applied with real signal model and a more detailed description of the algorithm search and functionality can be found from the references [17], [21].

IV. Performance Examples

The performance examples done via computer simulations are presented in this section. The effect of considered preprocessing methods to the performance and complexity of the LSD algorithms. In the computer simulations, a MIMO–OFDM system model was assumed with 512 subcarriers (300 used) according to the 3G long term evolution (LTE) parameters [22]. A bit interleaved coded modulation (BICM) with $1/2$ rate [13,15] turbo code was applied for the system. An uncorrelated (UNC) and highly correlated (CORR) typical

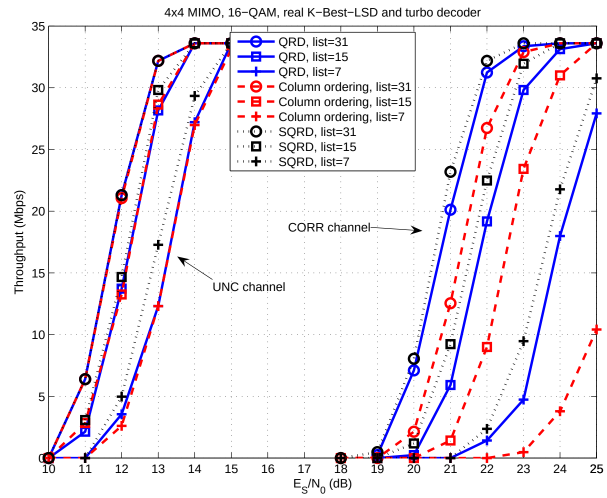

Fig. 3. Throughput vs SNR: Performance of the real K-best-LSD with different list sizes and preprocessing methods in 4×4 antenna system with 16-QAM.

urban (TU) 6 tap channels were assumed with a user velocity of 120 kmph. The system was operating with 5 MHz bandwidth at a carrier frequency of 2.4 GHz. The K-best-LSD, the SEE-LSD, and the IR-LSD were considered for detection and an iterative max-log-MAP turbo decoder with 8 iterations was used for decoding. Iterative detection and decoding was not assumed in the simulations. The K-best-LSD algorithm was applied with $C_0 = \infty$.

A. Simulation results

The performance of K-best-LSD was studied with different preprocessing algorithms and with different list sizes. The number of visited nodes by the K-best-LSD is fixed with given output list size K, and a higher K value results in better performance to certain extend as the LLR approximation gets better. The performance of the K-best-LSD with different list sizes in both UNC and CORR channels is shown in Figure 3. It can be seen that the when the applied list size is high enough in UNC channel, the performance difference between different preprocessing methods is not significant. When the list size is low enough or the channel is highly correlated, the SQRD algorithm ordering gives approximately 0.2 dB additional gain over the traditional QRD without ordering. The column ordering according to the Euclidean norm, however, actually shows worse performance compared to the other preprocessing methods in the CORR channel. The results indicate that the Euclidean norm of the channel matrix columns is not very good method to determine the detection order of the transmitted layers especially in a correlated channel realization. Also it can be noted that the additional gain by the SQRD algorithm is higher in CORR channel compared to the UNC channel.

The number of visited nodes by the sequential search LSD algorithms, the SEE-LSD and the IR-LSD, is a variable that depends on the channel realization. The total complexity of the LSD algorithms is relative to the number of visited nodes in

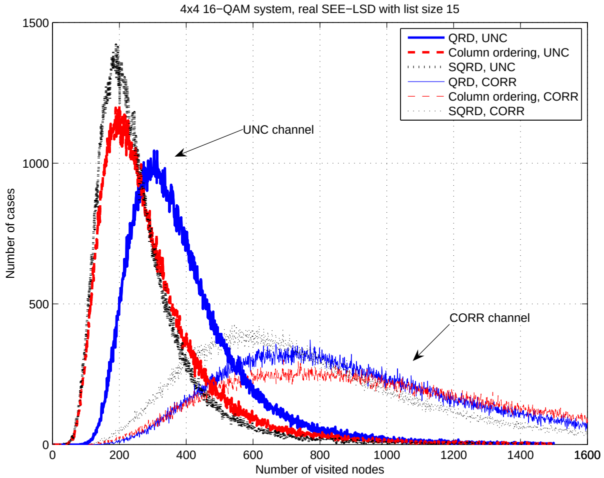

Fig. 4. A histogram of visited nodes by the SEE-LSD algorithm with different preprocessing methods.

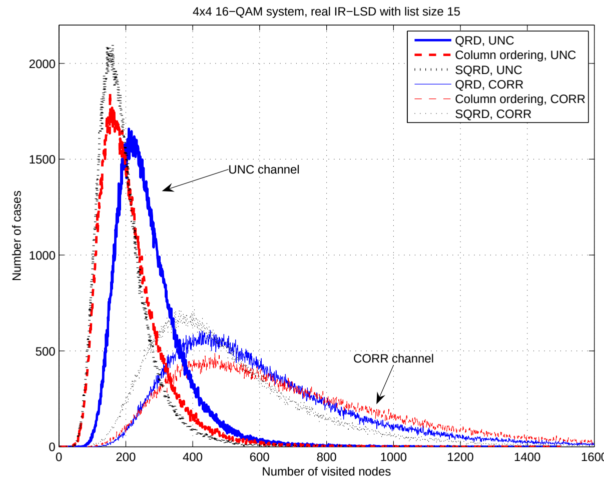

Fig. 5. A histogram of visited nodes by the IR-LSD algorithm with different preprocessing methods.

the search tree. Thus, we studied the distribution of the number of visited nodes by the LSD algorithms and the performance of the system with limited maximum number of visited nodes.

Histograms of the visited nodes by the SEE-LSD and IR-LSD algorithms with different preprocessing methods in UNC and CORR channel are shown in Figures 4 and 5. The average number of visited nodes by the algorithms with different preprocessing methods in both channel scenarios are listed in Table I. The ratio of visited nodes by the LSD with the column ordering and SQRD preprocessing compared to the traditional QRD preprocessing is shown in brackets. It can be seen that the correlation properties of the channel effect significantly to the number of visited nodes. Figures 4 and 5, and Table I show that both the column ordering according to the Euclidean norm and the SQRD decrease the distribution of the number of visited nodes clearly for the UNC channel approximately 20% and 30%, respectively. The results in CORR channel show, similarly as with K-best-LSD, that the column ordering according to the Euclidean norm actually increases the number of visited nodes by the SEE-LSD and the IR-LSD algorithms. The SQRD preprocessing, however, decreases the number of visited nodes approximately 20% on average compared to the QRD. We also studied the performance of the SEE-LSD algorithm with maximum number of visited nodes limited. The performance results for SEE-LSD and IR-LSD algorithms with maximum node limits are shown in Figure 6 and 7, respectively. The performance results also show that the column ordering according to the Euclidean norm is not feasible in CORR channel. An LSD with the SQRD preprocessing performs approximately $0.1-0.5$ dB better than an LSD with traditional QRD with the same maximum node limits.

B. Complexity comparisons

The simulations results showed that the SQRD as preprocessing decreases the number of required visited nodes by the LSD algorithms by approximately $20-30\%$. The illustrate the decrease in complexity the number of additional multiplication

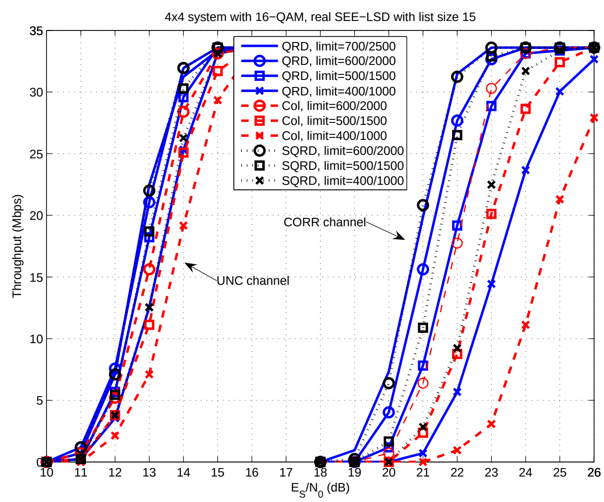

Fig. 6. Throughput vs SNR: Performance of the real SEE-LSD with different maximum node limits (UNC/CORR) and preprocessing methods in 4×4 antenna system with 16-QAM.

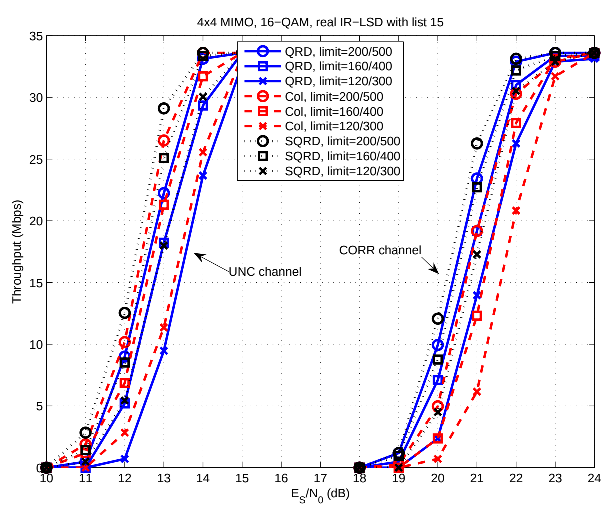

Fig. 7. Throughput vs SNR: Performance of the real IR-LSD with different maximum node limits (UNC/CORR) and preprocessing methods in 4×4 antenna system with 16-QAM.

(MUL) and addition (ADD) operations in preprocessing algorithms compared to traditional QRD are listed in Table II. The number operations required in the PED calculation in (4) are also listed in Table II assuming that the average current in PED calculation is the middle layer of the search tree. The numbers of additional and saved operations on average due to reduced number of visited nodes are listed for IR-LSD and SEE-LSD algorithms with SQRD preprocessing in 4×4 system with 16-QAM in Table III. It can be seen that a significant number of operations are saved with the SQRD applied as preprocessing.

V. Conclusions

We studied the effect of preprocessing to the complexity and performance of the LSD algorithms. We showed that the LSD algorithms benefit from proper ordering of the spatial layers prior detection. The study showed that the SQRD algorithm applied as the preprocessing is a simple way to decrease complexity of the LSD and it decreases the number of visited nodes approximately $20 - 30\%$ compared to the traditional QRD which results in significant number of saved arithmetic operations in the LSD. We also showed that the plain channel matrix column ordering is not feasible preprocessing method to be used with LSD in highly correlated channel realization.

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TABLE I
The average number of visited nodes by the SEE-LSD† and IR-LSD* algorithms in different channels with different preprocessing methods.

	UNC	CORR
QRD	$397^\dagger$ (100%) / 268^* (100%)	$1001^\dagger$ (100%) / 618^* (100%)
Col	$302^\dagger$ (76%) / 213^* (79%)	$1106^\dagger$ (110%) / 705^* (114%)
SQRD	$269^\dagger$ (68%) / 193^* (72%)	$821^\dagger$ (82%) / 509^* (82%)

TABLE II
The number of additional real operations due to preprocessing and the PED calculation operations in (4).

	Col	SQRD	PED
MUL	$N_T N_R$	$N_T N_R + \sum_{j=1}^{N_T-1} j$	$2N_T - 4 + 1$
ADD	$N_T(N_R - 1)$	$N_T(N_R - 1) + \sum_{j=1}^{N_T-1} j$	$2N_T - 4 + 1$

TABLE III
The number of added and saved real operations with the IR-LSD and the SEE-LSD with the SQRD in 4×4 system with 16-QAM and in UNC* and CORR† channels.

	IR-LSD (saved)	SEE-LSD (saved)	SQRD (additional)
MUL	$375^*/545^\dagger$	$640^*/900^\dagger$	22
ADD	$375^*/545^\dagger$	$640^*/900^\dagger$	18